LOOPS OF KINDNESS

AP CALC UNIT 2

Differentiation: Definition and Fundamental Properties

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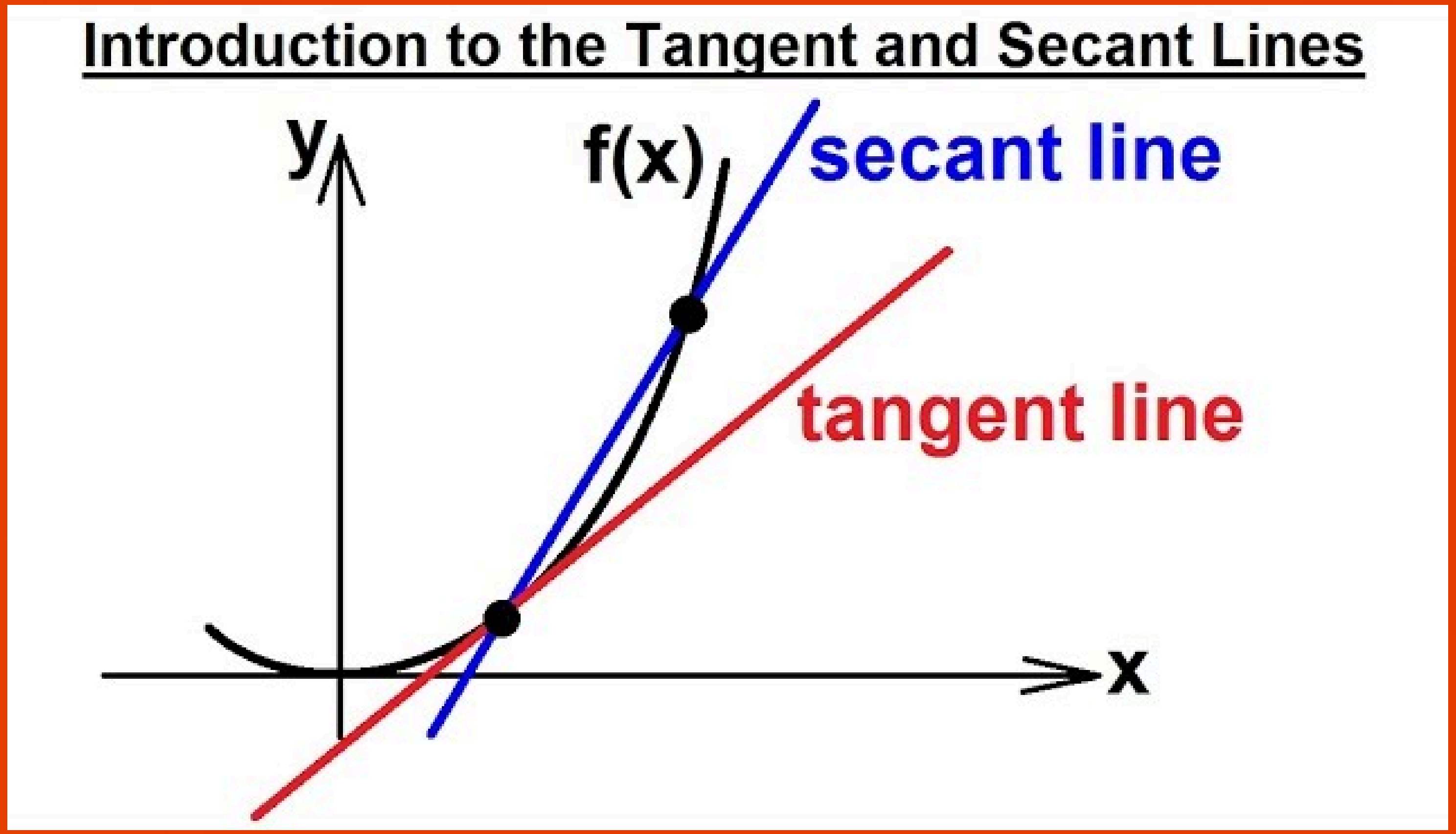
CONTENTS

Here are all the topics we will cover:

2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

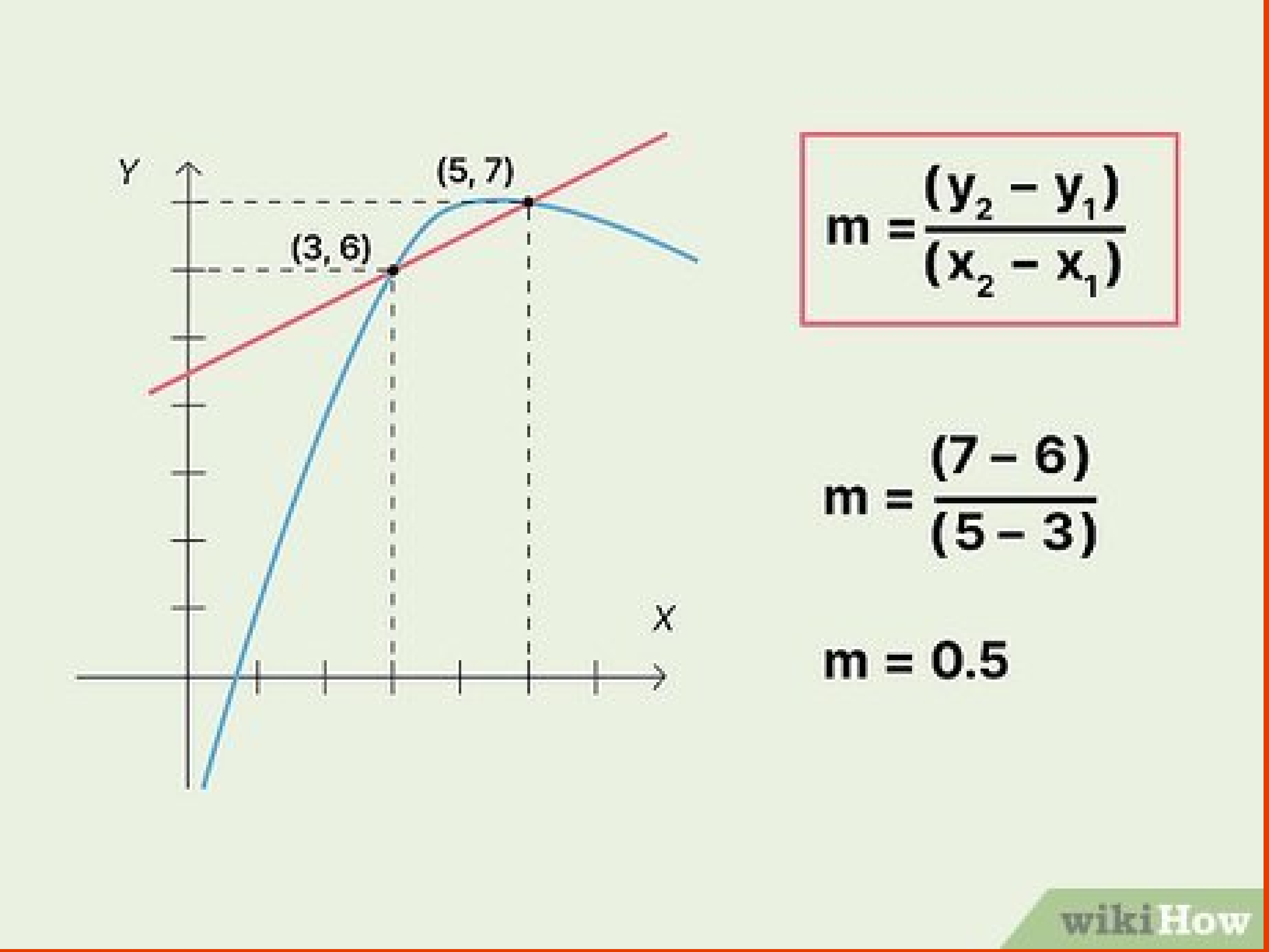
One of the most central problems in calculus is the tangent line problem. This problem centered around finding the slope of a tangent line. A tangent line is just a line that is touching a function at only 1 point.



A secant line is a line that touches a function at two points.

2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

The slope of a secant line is also known as the average rate of change. Since we have two points, we can just use our regular rise/run formula to get its slope, as in this example:

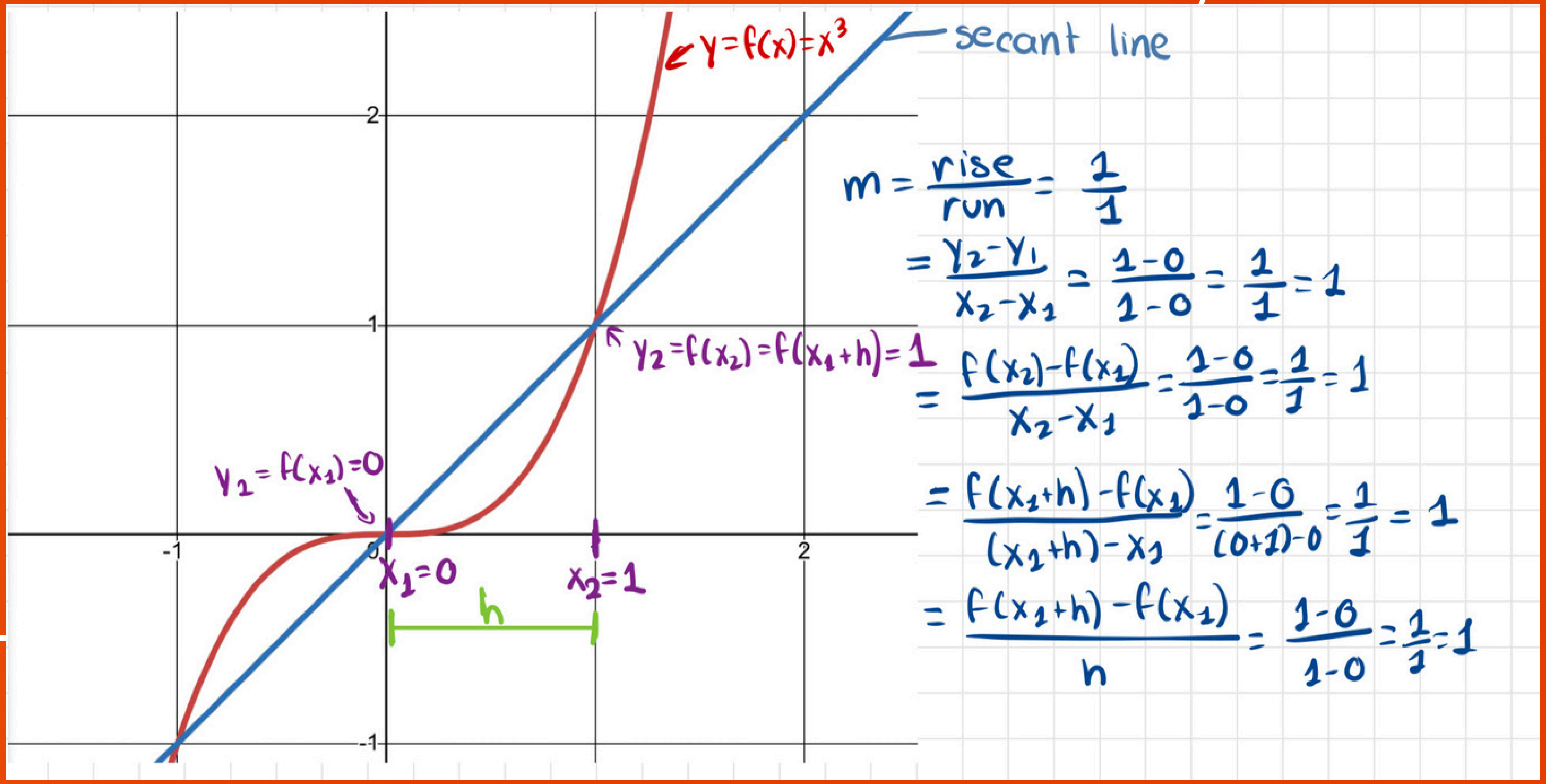


2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

We can rewrite it into a fancier formula. f $y=f(x)$ and the distance between x_1 and x_2 is h ($x_1 + h = x_2$), then we can rewrite the classic rise/run like this:

$$\frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1+h) - f(x_1)}{(x_1+h) - x_1} = \frac{f(x_1+h) - f(x_1)}{h}$$

You can visualize all these variables with a secant line of $y=x^3$ with $x_1 = 0$ and $x_2 = 1$



2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

This last fancy formula is known as a difference quotient! (difference just means subtraction, and quotient just means division.)

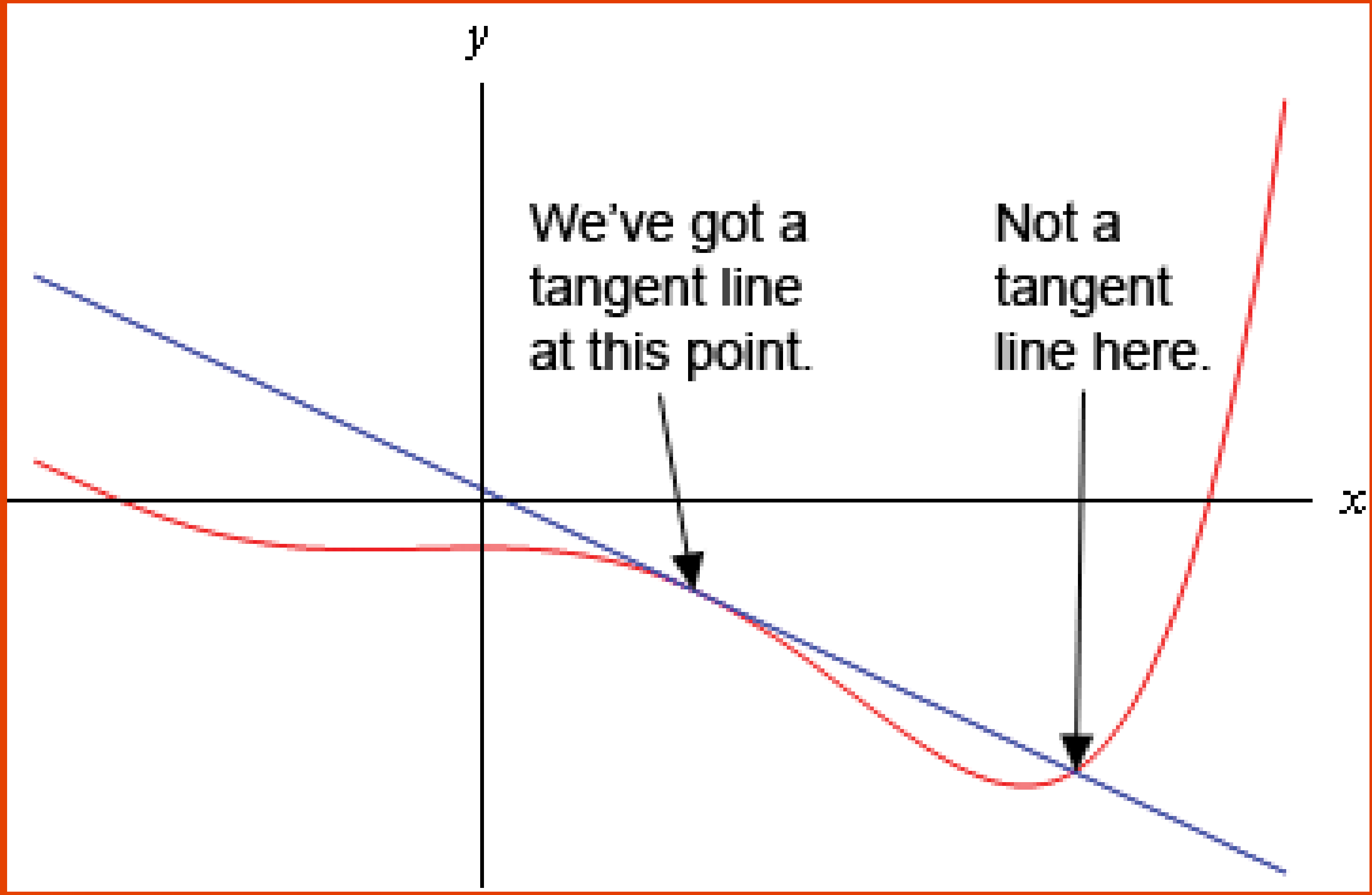
$$\frac{f(x_1+h)-f(x_1)}{h}$$

We usually express this with $x_1 = a$. This will come in handy later when we have to define rate of change at a point.

$$\frac{f(a+h)-f(a)}{h}$$

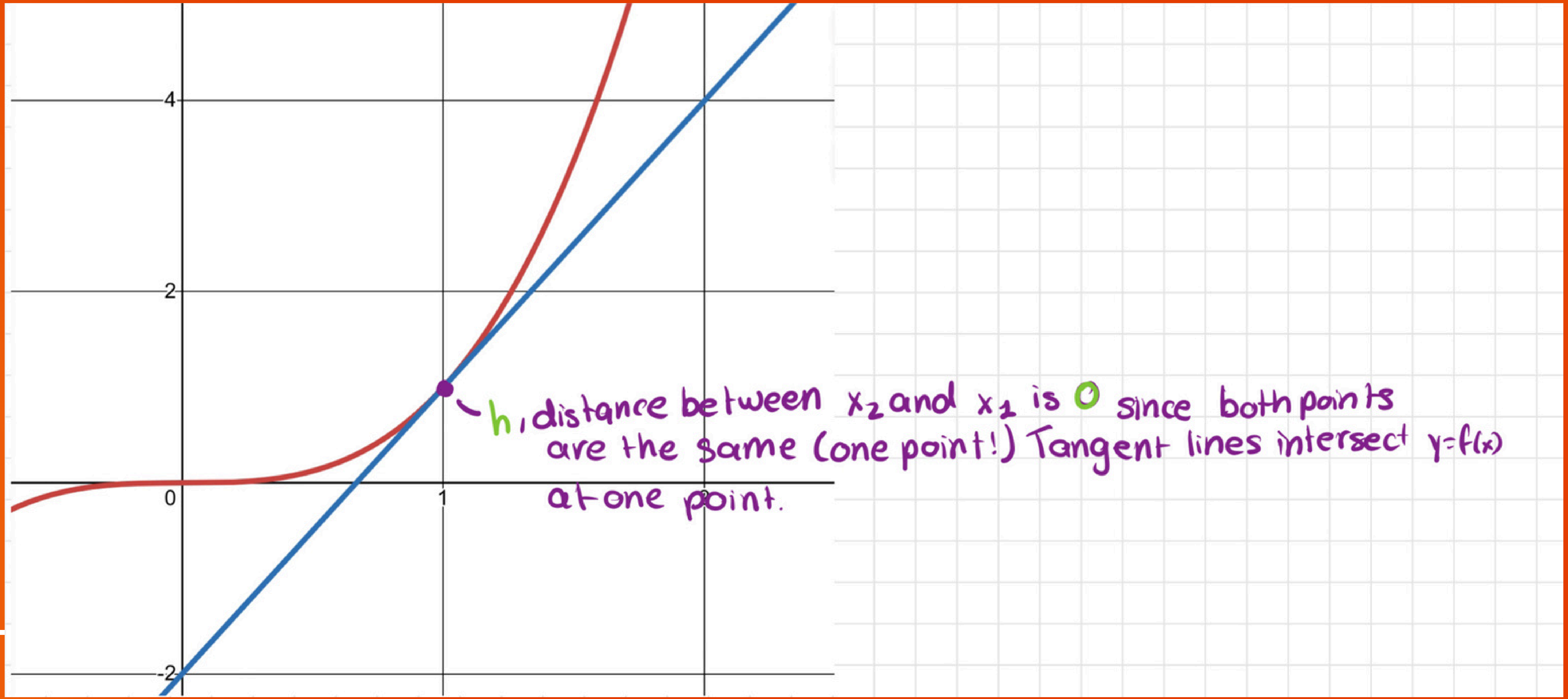
2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

Tangent lines are different. Since they only touch the curve at one point, they are known as the instantaneous rate of change. They only touch the function at one instance.



2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

Since a tangent line only touches a function at one point, we want the difference between the points, h, to be 0 (h = 0). If h = 0, then a + h = a + 0 = a, so the starting point and the ending points are both x = a . Since we only want to find the slope for one point, having the two points be the same seems like the solution!



2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

However, if we try just to plug in h = 0 into our difference quotient, something very sad happens. The equation is in indeterminate form (0/0), so we have no clue what it actually is!

$$\frac{f(a+0)-f(a)}{0} = \frac{f(a)-f(a)}{0} = \frac{0}{0}$$

We solve this by using limits to indicate that h is getting really, really close to 0, but it's never actually getting there. If h never makes it to 0, then we don't have to deal with indeterminate form.

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

Another way of expressing the difference quotient is by letting the first point be (a, f(a)) and the second point be (x, f(x)). If you use the standard rise/run formula, the slope of the line between the two points is:

$$\frac{f(x)-f(a)}{x-a}$$

This is another way of expressing the slope of a secant line, or the average rate of change.

Since the slope of the tangent line is reached when the two points get closer and closer until reaching one single point, we can express the slope of the tangent line as

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

This is equivalent to the slope of the tangent at x = a, since x gets closer and closer to a.

2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

What we've just found is the definition of a derivative. A derivative is just the slope of a function at a point, or the instantaneous rate of change. The derivative of f(x) at point x = a is denoted by

$$f'(a)$$

This is equivalent to the two limit forms of the difference quotient we've just found:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

What is the average rate of change of $g(x) = \dfrac{2}{x+3}$ over the interval $1 \leq x \leq 3$?

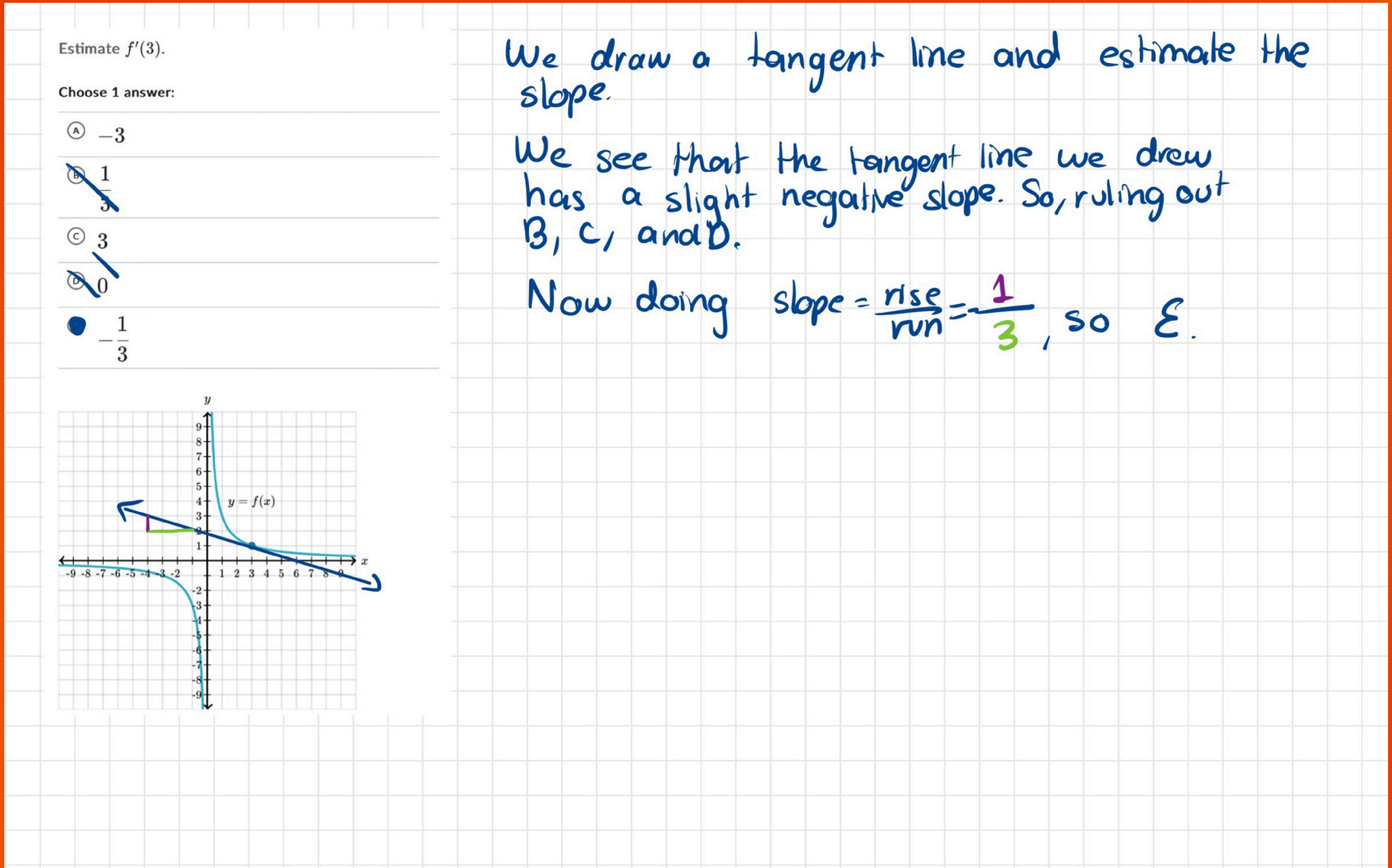
Average rate of change: $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

Our $x_2=3$ and $x_1=1$, so average rate of change $= \dfrac{f(3)-f(1)}{3-1} = \dfrac{\frac{2}{3+3}-\frac{2}{1+3}}{2}$

$$= \frac{\frac{2}{6}-\frac{2}{4}}{2} = \frac{\frac{1}{3}-\frac{1}{2}}{2} = \frac{\frac{2}{6}-\frac{3}{6}}{2}$$

$$= \frac{-\frac{1}{6}}{2} = \boxed{-\frac{1}{12}}$$

2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT



2.1 DEFINING AVERAGE AND INSTANTANEOUS RATES OF CHANGE AT A POINT

For a function f, we are given that $f(8) = 1$ and $f'(8) = 2$.

What's the equation of the tangent line to the graph of f at $x = 8$?

We have point-slope form $y - y_1 = m(x - x_1)$

We are given point $(x_1, y_2) = (8, 1)$

Since the slope of the tangent line is the same as the derivative, $f'(8) = 2$ means that the slope m of the tangent line is $m = 2$.

So, the equation of the tangent line in point-slope is

$$y - 1 = 2(x - 8)$$

2.2 DEFINING THE DERIVATIVE OF A FUNCTION AND USING DERIVATIVE NOTATION

As we learned in the previous lesson, a derivative is the instantaneous rate of change of a function. It is represented by this limit, provided it exists.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

2.2 DEFINING THE DERIVATIVE OF A FUNCTION AND USING DERIVATIVE NOTATION

In AP Calc, there are two main notations for the derivative. Here they are:

	Lagrange	Leibniz
Function	$f(x)$	f
First Derivative	$f'(x)$	$\frac{df}{dx}$

2.2 DEFINING THE DERIVATIVE OF A FUNCTION AND USING DERIVATIVE NOTATION

Which of the following is equal to $f'(4)$ for $f(x) = \sqrt{x}$?

We have $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Substituting into both forms:

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$$

$$f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - \sqrt{4}}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

2.3 ESTIMATING DERIVATIVES OF A FUNCTION AT A POINT

The derivative of a function at a point can be estimated from graphs or tables.

To estimate the derivative from a graph, try drawing a tangent line to the point and using the rise/run of the tangent line to find the slope. Since the slope of a tangent line is the derivative, this rise/run is the value of the derivative.

To estimate the derivative from a table, get the two closest points given in the table and find the slope between them. Since this is a slope between two points, this is the slope of a secant line. However, since it is the slope of a secant line between two points close to the point we want, it is an approximation of the slope of the tangent line.

2.3 ESTIMATING DERIVATIVES OF A FUNCTION AT A POINT

This table gives select values of the differentiable function h.

x	-4	-1	0	1	4
$h(x)$	-26	-15	-32	-39	-35

What is the best estimate for $h'(-2)$ we can make based on this table?

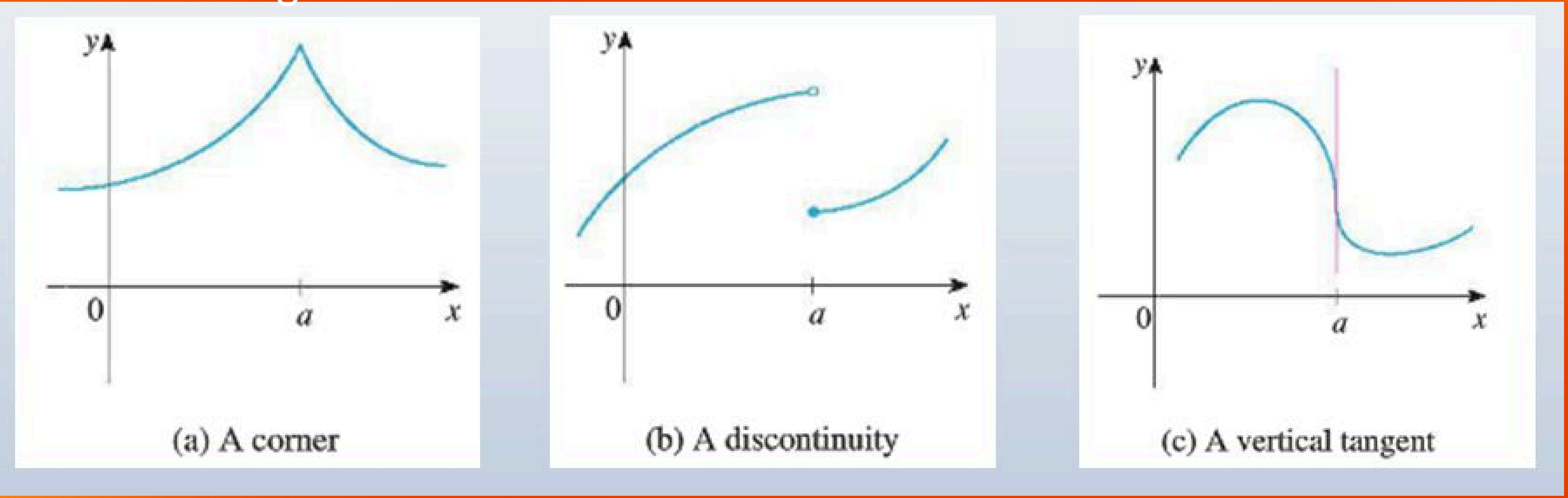
$$h'(-2) \approx \frac{h(-4)-h(-1)}{-4+1} = \frac{-26+15}{-3} = \frac{11}{3} \approx 3.67$$

two closest points that sandwich $x=-2$

2.4 CONNECTING DIFFERENTIABILITY AND CONTINUITY: DETERMINING WHEN DERIVATIVES DO AND DO NOT EXIST

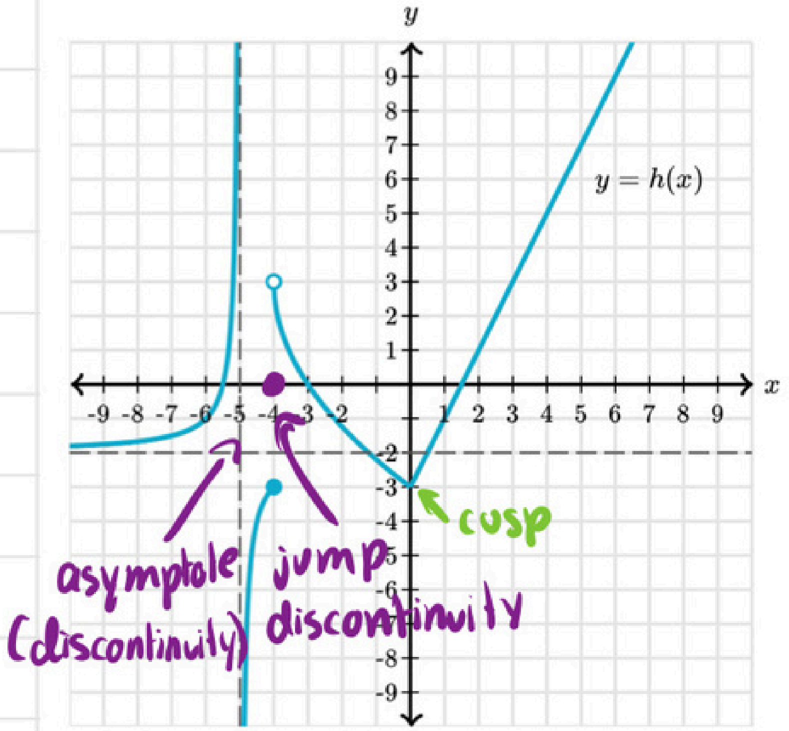
The derivative does not exist whenever

1. There is a discontinuity (could be jump, hole, anywhere the function doesn't exist)
2. Corner or cusp
3. Vertical tangent line



2.4 CONNECTING DIFFERENTIABILITY AND CONTINUITY: DETERMINING WHEN DERIVATIVES DO AND DO NOT EXIST

Function h is graphed. The dashed lines represent asymptotes.



Select all the x-values for which h is not differentiable.

h is not differentiable when there is a cusp, a discontinuity, or a vertical tangent.

Where are there discontinuities? Whenever you have to pick up the pen to continue drawing the function.

There is an asymptote at $x=-5$

A jump discontinuity at $x=-4$

Where are there cusps? Whenever you have to make a sharp turn.

There is a cusp at $x=0$

So, at $x=-5$, $x=-4$, $x=0$

2.5 THE POWER RULE

Power Rule Formula

$$\frac{d}{dx}(x^n) = n.x^{n-1}$$

EXAMPLE

$$\frac{d}{dx}\left[x^2\right] = 2x^{(2-1)} = 2x^1 = 2x$$

2.5 THE POWER RULE

Let $h(x) = \sqrt{x^5}$.

$h'(4) =$ 20

Rewriting $h(x) = \sqrt{x^5}$ into exponential form,

$h(x) = x^{5/2}$

Using power rule, $h'(x) = \frac{5}{2}x^{\frac{5}{2}-\frac{2}{2}} = \frac{5}{2}x^{\frac{3}{2}}$

So, $h'(4) = \frac{5}{2}(4)^{\frac{3}{2}} = \frac{5}{2}(2)^3 = 5(2)^2 = 5(4) = 20$

2.6 DERIVATIVE RULES: CONSTANT, SUM, DIFFERENCE, AND CONSTANT MULTIPLE

CONSTANT

$$\frac{d}{dx}(c) = 0$$

CONSTANT MULTIPLE

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

SUM/DIFFERENCE

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

2.6 DERIVATIVE RULES: CONSTANT, SUM, DIFFERENCE, AND CONSTANT MULTIPLE

Let $h(x) = \frac{-4}{x^3} + \frac{1}{x}$.

$h'(-2) =$ 1/2

sum rule — constant multiple

$$h'(x) = \frac{d}{dx}\left(-\frac{4}{x^3}\right) + \frac{d}{dx}\left(\frac{1}{x}\right) = -4\frac{d}{dx}\left(\frac{1}{x^3}\right) + \frac{d}{dx}\left(\frac{1}{x}\right) = -4\frac{d}{dx}\left(x^{-3}\right) + \frac{d}{dx}\left(x^{-1}\right)$$

$$= -4\left(-3x^{-3-1}\right) + \left(-1x^{-1-1}\right) = 12x^{-4} - x^{-2} = \frac{12}{x^4} - \frac{1}{x^2}$$

power rule — solve

$$h'(-2) = \frac{12}{(-2)^4} - \frac{1}{(-2)^2} = \frac{12}{16} - \frac{1}{4} = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

2.7 DERIVATIVES OF COS, SIN, EXPONENTIAL, LOGARITHMIC FUNCTIONS

$$\frac{d}{dx}\cos x = \sin x$$

$$\frac{d}{dx}\sin x = -\cos x$$

$$\frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\log_a x = \frac{1}{x} \cdot \frac{1}{\ln a}$$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\log_e x = \frac{1}{x}$$

2.8 THE PRODUCT RULE

The Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

The way I remember this is by remembering "one no, one yes, flipped." This reminds me to not take the derivative of the first function, then take take the derivative of the second function, and then add the "flipped" version.

2.8 THE PRODUCT RULE

Let $h(x) = \ln(x)\cos(x)$.

$h'(x) = \square$

$$h'(x) = \ln(x)\left(\cos(x)\right)' + \cos(x)\left(\ln(x)\right)'$$

$$= \ln(x)\left(-\sin(x)\right) + \cos(x)\left(\frac{1}{x}\right)$$

$$= \boxed{-\ln(x)\sin(x) + \frac{\cos(x)}{x}}$$

2.8 THE QUOTIENT RULE

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\left(g(x)\right)^2}$$

Remember this using "low d high, high d low, low low." This reminds you to get the denominator ("low" part of the fraction), multiply it by the derivative of the numerator ("high" part of the fraction), subtract the high times the derivative of the low, all over low^2 ("low low").

2.10 DERIVATIVES OF TAN(X), COT(X), SEC(X), AND CSC(X)

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

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